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Time Series Analysis via Rank Order Theory: Signed-Rank Tests for ARMA Models

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An asymptotic distribution theory is developed for a general class of signed-rank serial statistics, and is then used to derive asymptotically locally optimal tests (in the maximin sense) for testing an ARMA model against other ARMA models. Special cases yield Fisher–Yates, van der Waerden, and Wilcoxon type tests. The asymptotic relative efficiencies of the proposed procedures with respect to each other, and with respect to their normal theory counterparts, are provided.

1. INTRODUCTION

1.1. *Invariance, Ranks and Signed-Ranks*

Invariance arguments constitute the theoretical cornerstone of rank-based methods: whenever weaker unbiasedness or similarity properties can be considered satisfying or adequate, *permutation tests*, which generally follow from the usual *Neyman structure* arguments, are theoretically preferable to rank tests, since they are less restrictive and thus allow for more powerful results (though, of course, their practical implementation may be more problematic).

Which type of ranks (*signed* or *unsigned*) should be adopted—if

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rank-based techniques, hence invariance properties are to be considered—depends on the invariance features of the testing problem and maximal invariants at hand. If, for instance, the observations (or some function thereof) under the null hypothesis to be tested are i.i.d. with unspecified density, it is well known that the vector of (ordinary) ranks constitutes a maximal invariant (for the group of continuous, order-preserving transformations). If the common unspecified density further can be assumed symmetric with respect to the origin or any other specified median (call this the homogeneous symmetry hypothesis), signed ranks are maximal invariant (for the subgroup of continuous, order-preserving, *odd* transformations). If the assumption of identical distributions is dropped (call this the non-homogeneous symmetry hypothesis) then a maximal invariant is the vector of signs, etc. Clearly, ordinary ranks and the vector of signs both remain invariant under the symmetric i.i.d. case, but they no longer are *maximal* invariant, and using a rank test or a sign test instead of a signed-rank one generally results in a loss of relevant information, hence a loss of power. This loss can be dramatic (e.g., when using ranks in testing against a shift alternative) or it may be nil (e.g., when using signs in testing against double exponential shifts). One nevertheless always should conclude in favor of a maximal invariant, whenever a choice is possible.

Now, the effectiveness of the choice between signed and unsigned ranks, in the symmetric i.i.d. case, is somewhat obscured, in practice, by the fact that (unsigned) ranks are totally insensitive to a variety of alternatives. Testing the slope of a regression line, e.g., can be achieved (in a strictly unbiased manner) through either ordinary ranks or signed ones, whereas testing the intercept using ordinary ranks is impossible. A rather perverse consequence of this fact is that the type of alternative at hand, in most cases, apparently dictates which type of ranks (ordinary ranks for the slope, signed ranks for the intercept) should be adopted, and thus which invariance argument should be invoked—see, e.g., Puri and Sen [38, Sections 5.2 and 5.3] for a typical example of such a questionable attitude.

Serial dependence problems, such as that of testing an ARMA model with unspecified innovation density provide an interesting example where the choice between signed and unsigned ranks cannot be eluded and where power considerations do not supersede the much more fundamental invariance principles underlying this choice.

For a review of rank-based techniques in the time series analysis context, we refer to Dufour *et al.* [10], Bhattacharyya [2], and Hallin and Puri [23].

1.2. *Outline of the Paper*

Optimal rank-based tests for serial dependence problems were derived in Hallin *et al.* [17, 18] and Hallin and Puri [22]. Our objective here is to obtain signed-rank analogues of these optimal procedures. Indeed, it is often the case, in practice, that the innovation density of a series under study can be assumed to be symmetrical with respect to some known median—zero in most cases. This is the situation considered in several classical quick tests for randomness [41, 32, 14, 15], as well as in more recent papers (Dufour [6],—where the ranks are those of the products $X_t X_{t-1}$ of successive observations—or Tran [39], where *nonserial* signed-rank statistics are considered under various mixing conditions). As emphasized in Section 1.1, invariance considerations then lead to signed-rank techniques instead of unsigned-rank ones.

Accordingly, we develop here the asymptotic distribution theory required for constructing signed-rank tests based on a general class of *linear serial signed-rank statistics* (Section 2). We therefore establish the asymptotic equivalence of such statistics with adequate parametric ones, the asymptotic distribution of which we then obtain under the null hypothesis of homogeneous symmetry, as well as under local alternatives of second-order linear dependence. A special case of serial signed-rank statistic—the so-called signed-rank autocorrelation coefficients—is introduced in Section 3. In Section 4, we show that asymptotically optimal signed-rank tests for the problem of testing an ARMA model with unspecified, symmetric innovation density can be obtained on substituting these signed autocorrelations for the unsigned ones in the optimal test statistics described in Hallin and Puri [22]. Special cases yield statistics of the *Fisher–Yates–Fraser*, *van der Waerden*, *Wilcoxon*, or *Laplace* types, which are asymptotically equivalent (up to $o_P(n^{-1/2})$ terms) under the hypothesis of homogeneous symmetry as well as under local alternatives, to their unsigned counterparts: the power comparisons and asymptotic relative efficiencies discussed in Hallin and Puri [22] and Hallin and Mélard [20] thus remain valid here.

The fact that their mutual AREs are one, however, does not imply that the advantage of using signed-rank autocorrelations instead of unsigned ones is nil, or negligible, either for short series lengths, or even asymptotically. Numerical investigations of the performances of signed-rank tests [21] indicate that, for fixed values of n, the power of signed-rank procedures can be substantially larger than that of unsigned ones, an empirical finding that should be confirmed by a theoretical investigation of the corresponding deficiencies.

Because of their ease of application, excellent overall performance or broader invariance and distribution-freeness properties, one may also be interested in non-optimal testing procedures, such as the signed version of the Spearman–Wald–Wolfowitz autocorrelation coefficient (see [40 or 1]), or the generalized runs tests considered in Dufour and Hallin [8]. The AREs of such tests can be obtained from Proposition 2.2. Explicit ARE values (with respect to normal-theory procedures) are provided, which for these generalized runs tests, are unexpectedly high. A numerical application is discussed in Section 5.

2. Asymptotic Distribution Theory of Linear Serial Signed-Rank Statistics

2.1. *Linear Serial Signed-Rank Statistics*

Consider a series $\mathbf{Z}^{(n)} = (Z_1^{(n)}, ..., Z_t^{(n)}, ..., Z_n^{(n)})$ of length n; depending on the particular problem at hand, $\mathbf{Z}^{(n)}$ will be the observed series $\mathbf{X}^{(n)} = (X_1^{(n)}, ..., X_t^{(n)}, ..., X_n^{(n)})$ itself, or it will result from $\mathbf{X}^{(n)}$ by means of some adequate transformation. Denote by $R_{+,t}^{(n)}$ the rank of $|Z_t^{(n)}|$ among the absolute values $|Z_1^{(n)}|, ..., |Z_n^{(n)}|$, by $\mathbf{R}_+^{(n)}$ the vector of ranks $(R_{+,1}^{(n)}, ..., R_{+,n}^{(n)})$, by $\text{sgn}(Z_t^{(n)})$ the sign of $Z_t^{(n)}$ and by $\mathbf{sgn}(\mathbf{Z}^{(n)})$ the vector of signs $(\text{sgn}(Z_1^{(n)}), ..., \text{sgn}(Z_n^{(n)}))$. Under the null hypothesis $H_0^{(n)}$ that $Z_1^{(n)}, ..., Z_n^{(n)}$ are independently and identically distributed, according to some absolutely continuous distribution function with unspecified density $f(x)$ satisfying $f(x) = f(-x)$, $\mathbf{R}_+^{(n)}$ constitutes (with probability one) a random permutation of $\{1, ..., n\}$, $\mathbf{sgn}(\mathbf{Z}^{(n)})$ is uniformly distributed on $\{-1, 1\}^n$ (still with probability one), and the two vectors $\mathbf{R}_+^{(n)}$ and $\mathbf{sgn}(\mathbf{Z}^{(n)})$ are mutually independent. $H_0^{(n)}$ is usually known as the hypothesis of (homogeneous) symmetry although in the present context, randomness will be emphasized rather than symmetry. This same hypothesis will be denoted by $H_f^{(n)}$ whenever the density f is specified (up to a scale parameter). Both $\mathbf{R}_+^{(n)}$ and $\mathbf{sgn}(\mathbf{Z}^{(n)})$ are indeed invariant with respect to scale transformations.

Letting $a_+^{(n)}(i_1, ..., i_{k+1})$, where $\{i_1, ..., i_{k+1}\}$ is a $(k+1)$-tuple of elements of $\{\pm 1, \pm 2, ..., \pm n\}$ such that all absolute values $|i_1|, ..., |i_{k+1}|$ are distinct, denote a set of $2^{k+1}n!/(n-k-1)!$ *scores*, define a *linear serial signed-rank statistic* of order k as a statistic of the form

$$S_+^{(n)} = (n-k)^{-1} \sum_{t=k+1}^{n} a_+^{(n)}(\text{sgn}(Z_t^{(n)})\, R_{+,t}^{(n)}, \text{sgn}(Z_{t-1}^{(n)})\, R_{+,t-1}^{(n)}, ..., \text{sgn}(Z_{t-k}^{(n)})\, R_{+,t-k}^{(n)}). \tag{2.1}$$

Classical examples are

(i) the *classical runs test statistic* (with runs defined with respect to zero), of order one, characterized by

$$a_+^{(n)}(i_1, i_2) = \begin{cases} 1 & \text{if } \; i_1 i_2 < 0 \\ 0 & \text{if } \; i_1 i_2 \geqslant 0. \end{cases} \tag{2.2}$$

The resulting test is Goodman's [14] *simplified runs test* (with constant $k = 0$—see also [15]), which coincides with Dufour's [6] runs test (although the latter is described in terms of the ranks of the products $|Z_t^{(n)} Z_{t-1}^{(n)}|$).

(ii) the *turning point* or *runs up and down test statistic*, of order two, with scores

$$a_+^{(n)}(i_1, i_2, i_3) = \begin{cases} 1 & \text{if } \ i_1 > i_2 < i_3 \quad \text{or} \quad i_1 < i_2 > i_3 \\ 0 & \text{elsewhere.} \end{cases} \tag{2.3}$$

This test statistic is also measurable with respect to the usual (un-signed) ranks, and was studied by Wallis and Moore [41], Moore and Wallis [32], and Levene [28].

Both the runs test and the turning point test have very desirable properties of quickness, ease of use, and robustness. However, being *quick* tests, they cannot be hoped to be very powerful (see [15] for a study of the power function under Markovian alternatives and Sections 4.2 and 5 below for exact ARE values and numerical examples). Better results can be expected from, e.g.,

(iii) the signed version of the *Spearman–Wald–Wolfowitz rank serial correlation coefficient*, of order $k \geqslant 1$—call it the *signed Spearman autocorrelation of order k*—with scores (up to multiplicative and additive constants)

$$a_+^{(n)}(i_1, i_2, ..., i_{k+1}) = i_1 i_{k+1}. \tag{2.4}$$

This signed version, to the best of our knowledge, has not been considered so far. Its superiority over its unsigned counterpart [40, 1] is attested in Section 5 below.

Locally asymptotically optimal procedures for time-series analysis testing problems, however, typically involve signed versions of the *f-rank autocorrelation coefficients* introduced in Hallin *et al.* [18] and Hallin and Puri [22]. The *signed f-rank autocorrelation of order k* is characterized (up to multiplicative and additive constants—see (3.1) and (3.3) for a more precise definition) by scores $a_+^{(n)}(i_1, i_2, ..., i_{k+1})$ which are proportional to

$$\phi[F^{-1}[(n+1+i_1)/2(n+1)]]\, F^{-1}[(n+1+i_{k+1})/2(n+1)], \tag{2.5}$$

where $f(z)$, $F(z)$, and $\phi(z)$ denote a symmetric probability density function (satisfying a few regularity assumptions, such as being derivable—see Section 2.2), the corresponding distribution function, and the logarithmic derivative $-(d/dz)\log f(z)$, respectively. An alternative, asymptotically equivalent, version is obtained by replacing (2.5) with

$$E\{\phi[F^{-1}[(1+\operatorname{sgn}(i_1)\,U_n^{(|i_1|)})/2]]\}\,E\{F^{-1}[(1+\operatorname{sgn}(i_{k+1})\,U_n^{(|i_{k+1}|)})/2]]\}, \tag{2.6}$$

where $U_n^{(1)} \leqslant U_n^{(2)} \leqslant \cdots \leqslant U_n^{(n)}$ are the ordered random variables of a sample of size n from the rectangular $[0, 1]$ distribution function.

Let also $V_1, ..., V_n$ denote an n-tuple of i.i.d. rectangular $[-1, +1]$ variables. Considering a signed-rank statistic $S_+^{(n)}$ of order k, with scores $a_+^{(n)}(\cdots)$, assume the existence of a function J_+, defined on the $(k+1)$-dimensional open square $(-1, 1)^{k+1}$, such that for some strictly positive δ

$$E[|J_+(V_1, V_2, ..., V_{k+1})|^{2+\delta}] < \infty. \tag{2.7}$$

Assume, furthermore, that

$$\lim_{n\to\infty} E\{[a_+^{(n)}(\operatorname{sgn}(V_1)\,R_{+,1}^{(n)}, ..., \operatorname{sgn}(V_{k+1})\,R_{+,k+1}^{(n)}) \\ - J_+(V_1, ..., V_{k+1})]^2\} = 0, \tag{2.8}$$

where $R_{+,i}^{(n)}$ denotes the rank of $|V_i|$ among $|V_1|, ..., |V_n|$. Such a function $J_+(\cdots)$ is called a *score-generating function* for $S_+^{(n)}$.

Associated with $J_+(\cdots)$, define the *normalized* score-generating function $J_+^*(\cdots)$ as

$$\begin{aligned} J_+^*&(v_1, v_2, ..., v_{k+1}) \\ &= J_+(v_1, ..., v_{k+1}) - \sum_{l=1}^{k+1} E[J_+(V_1, ..., V_{k+1})\,|\,|V_l| = |v_1|] \\ &\quad + kE[J_+(V_1, ..., V_{k+1})] \\ &= J_+(v_1, v_2, ..., v_{k+1}) \\ &\quad - 2^{-(k+1)} \sum_{s\in\{-1,1\}} \sum_{l=0}^{k} \int_{[-1,1]^k} J_+(w_1, ..., w_l, sv_1, w_{l+1}, ..., w_k)\,d\mathbf{w} \\ &\quad + 2^{-(k+1)}k \int_{[-1,1]^{k+1}} J_+(w_1, w_2, ..., w_{k+1})\,d\mathbf{w}, \end{aligned} \tag{2.9}$$

where $\int_{[-1,1]^k}(\cdots)\,d\mathbf{v}$ stands for the k-fold integral $\int_{-1}^{1}\cdots\int_{-1}^{1}(\cdots)$ $\operatorname{sgn}(v_1)\cdots\operatorname{sgn}(v_k)\,dv_1\cdots dv_k$.

Finally, denote by F the distribution function associated with the symmetric density function f, and put $F_+ = 2F - 1$. Clearly, under $H_f^{(n)}$, $F_+(Z_1^{(n)}), ..., F_+(Z_n^{(n)})$ are i.i.d. rectangular $[-1, +1]$ random variables. Also note that $\operatorname{sgn}(x)\,F_+(|x|) = F_+(x)$, and $F_+^{-1}(v) =$

$F^{-1}((v+1)/2)$, $v \in (-1, 1)$. Here, as in the sequel, the inverse H^{-1} of a nondecreasing function H is taken as $H^{-1}(v) = \inf\{x \mid H(x) \geqslant v\}$.

2.2. *Asymptotic Distribution of Linear Serial Signed-Rank Statistics*

The asymptotic distribution of a linear serial signed-rank statistic $S_+^{(n)}$ with score-generating function J_+ is provided here under the null hypothesis $H_0^{(n)}$ as well as under contiguous alternatives of serial dependence. Although the results of Hallin *et al.* [17] do not apply (the statistics $\mathscr{S}_+^{(n)}$ and $\mathscr{E}_+^{(n)}$ below are not of the same type as $\mathscr{S}^{(n)}$ and $\mathscr{E}^{(n)}$ in this latter reference), the technical ideas used in the proofs are essentially similar, and, except for Proposition 2.1, the details are left to the reader.

Define

$$\mathscr{S}_+^{(n)} = (n-k)^{-1} \sum_{t=k+1}^{n} J_+(F_+(Z_t^{(n)}), ..., F_+(Z_{t-k}^{(n)})) \tag{2.10}$$

and

$$\mathscr{E}_+^{(n)} = [2^{k+1}n(n-1) \cdot \cdots \cdot (n-k)]^{-1} \sum_{\mathbf{s} \in \{-1, 1\}^{k+1}} \sum_{1 \leqslant t_1 \neq \cdots \neq t_{k+1} \leqslant n} \cdots \sum A(\mathbf{s}, F_+)$$

where

$$A(\mathbf{s}, F_+) = J_+(s_1 F_+(Z_{t_1}^{(n)}), ..., s_{k+1} F_+(Z_{t_{k+1}}^{(n)})). \tag{2.11}$$

Note that $\mathscr{E}_+^{(n)}$ actually constitutes the mean of $\mathscr{S}_+^{(n)}$, conditional upon the order statistic of the series of absolute values $|Z_1^{(n)}|, ..., |Z_n^{(n)}|$. Finally, denote by $m_+^{(n)}$ the mean of $S_+^{(n)}$ under $H_0^{(n)}$:

$$m_+^{(n)} = [2^{k+1}n(n-1) \cdot \cdots \cdot (n-k)]^{-1} \times \sum_{\mathbf{s} \in \{-1, 1\}^{k+1}} \sum_{1 \leqslant i_k \neq \cdots \neq i_{k+1} \leqslant n} \cdots \sum a_+^{(n)}(s_1 i_1, ..., s_{k+1} i_{k+1}). \tag{2.12}$$

PROPOSITION 2.1. *Under* $H_f^{(n)}$, $(S_+^{(n)} - m_+^{(n)}) - (\mathscr{S}_+^{(n)} - \mathscr{E}_+^{(n)}) = o_p(n^{-1/2})$, *as* $n \to \infty$.

Proof. Let $\Delta^{(n)} = (n-k)^{1/2} (S_+^{(n)} - m_+^{(n)} - \mathscr{S}_+^{(n)} + \mathscr{E}_+^{(n)})$, and denote by $|\mathbf{Z}_{(\cdot)}^{(n)}| = (|Z^{(n)}|_{(1)}, ..., |Z^{(n)}|_{(n)})$ the order statistic for the series of absolute values $|Z_1^{(n)}|, ..., |Z_n^{(n)}|$. Under $H_f^{(n)}$,

$$\begin{aligned} E[(\Delta^{(n)})^2] &= E\{E\{[(n-k)^{1/2} (S_+^{(n)} - \mathscr{S}_+^{(n)}) \\ &\quad - (n-k)^{1/2} (m_+^{(n)} - \mathscr{E}_+^{(n)})]^2 \mid |\mathbf{Z}_{(\cdot)}^{(n)}|\}\} \\ &= (n-k)\, E\{D^2(S_+^{(n)} - \mathscr{S}_+^{(n)} \mid |\mathbf{Z}_{(\cdot)}^{(n)}|)\}, \end{aligned}$$

where $D^2(S_+^{(n)} - \mathscr{S}_+^{(n)} \mid |\mathbf{Z}_{(\cdot)}^{(n)}|)$ stands for the variance of $S_+^{(n)} - \mathscr{S}_+^{(n)}$ conditional upon $|\mathbf{Z}_{(\cdot)}^{(n)}|$. Denote by $(t(1), ..., t(n))$ the vector of *antiranks*—namely, the vector whose ith component $t(i)$ is such that $R^{(n)}_{+,t(i)} = i$, and by $\mathbf{s}^{(n)}_{(\cdot)}$ the vector with ith component $s^{(n)}_{(i)} = \operatorname{sgn}(Z^{(n)}_{t(i)})$. Then

$$\begin{aligned} D^2(S^{(n)}_+ - \mathscr{S}^{(n)}_+ \mid |\mathbf{Z}^{(n)}_{(\cdot)}|) &= E\{D^2[S^{(n)}_+ - \mathscr{S}^{(n)}_+ \mid \mathbf{s}^{(n)}_{(\cdot)}, |\mathbf{Z}^{(n)}_{(\cdot)}|] \mid |\mathbf{Z}^{(n)}_{(\cdot)}|\} \\ &\quad + D^2[E[S^{(n)}_+ - \mathscr{S}^{(n)}_+ \mid \mathbf{s}^{(n)}_{(\cdot)}, |\mathbf{Z}^{(n)}_{(\cdot)}|] \mid |\mathbf{Z}^{(n)}_{(\cdot)}|] \\ &= D^2_{(1)} + D^2_{(2)}, \qquad \text{say.} \end{aligned}$$

Conditional upon $|\mathbf{Z}^{(n)}_{(\cdot)}|$ and $\mathbf{s}^{(n)}_{(\cdot)}$, $S^{(n)}_+ - \mathscr{S}^{(n)}_+$ is a linear serial rank statistic of order k (with respect to the ranks $R^{(n)}_{+,t}$), with scores

$$\begin{aligned} A^{(n)}_{\mathbf{s}(\cdot), |\mathbf{Z}|}&(i_1, i_2, ..., i_{k+1}) \\ &= a^{(n)}_+(s_{(i_1)} i_1, ..., s_{(i_{k+1})} i_{k+1}) \\ &\quad - J_+(s_{(i_1)}(2F(|Z^{(n)}|_{(i_1)}) - 1), ..., s_{(i_{k+1})}(2F(|Z^{(n)}|_{(i_{k+1})}) - 1)). \end{aligned}$$

Using Lemmas 2 and 4 of Hallin *et al.* [17], $D^2_{(1)}$ is bounded by

$$\begin{aligned} (n-k)^{-1} & KE\{E[[A^{(n)}_{\mathbf{s}(\cdot), |\mathbf{Z}|}(R^{(n)}_{+,1}, ..., R^{(n)}_{+,k+1})]^2 \mid \mathbf{s}^{(n)}_{(\cdot)}, |\mathbf{Z}^{(n)}_{(\cdot)}|] \mid |\mathbf{Z}^{(n)}_{(\cdot)}|\} \\ &= (n-k)^{-1} KE\{[A^{(n)}_{\mathbf{s}(\cdot), |\mathbf{Z}|}(R^{(n)}_{+,1}, ..., R^{(n)}_{+,k+1})]^2 \mid |\mathbf{Z}^{(n)}_{(\cdot)}|\} \end{aligned}$$

(still under $H^{(n)}_f$), where K denotes a constant. Turning to $D^2_{(2)}$,

$$\begin{aligned} E[S^{(n)}_+ &- \mathscr{S}^{(n)}_+ \mid \mathbf{s}^{(n)}_{(\cdot)}, |\mathbf{Z}^{(n)}_{(\cdot)}|] \\ &= [n(n-1) \cdot \cdots \cdot (n-k)]^{-1} \sum_{1 \leqslant t_1 \neq \cdots \neq t_{k+1} \leqslant n} \cdots \sum A^{(n)}_{\mathbf{s}(\cdot), |\mathbf{Z}|}(t_1, ..., t_{k+1}). \end{aligned}$$

Hence,

$$\begin{aligned} D^2_{(2)} &= [n(n-1) \cdot \cdots \cdot (n-k)]^{-2} \\ &\quad \times \sum_{1 \leqslant t_1 \neq \cdots \neq t_{k+1} \leqslant n} \cdots \sum \Bigg[D^2(A^{(n)}_{\mathbf{s}(\cdot), |\mathbf{Z}|}(t_1, ..., t_{k+1}) \mid |\mathbf{Z}^{(n)}_{(\cdot)}|) \\ &\quad + \sum_{1 \leqslant t'_1 \neq \cdots \neq t'_{k+1} \leqslant n} \cdots \sum \operatorname{Cov}(A^{(n)}_{\mathbf{s}(\cdot), |\mathbf{Z}|}(t_1, ..., t_{k+1}), A^{(n)}_{\mathbf{s}(\cdot), |\mathbf{Z}|}(t'_1, ..., t'_{k+1}) \mid |\mathbf{Z}^{(n)}_{(\cdot)}|) \Bigg], \end{aligned}$$

where the summation over $t'_1 \cdots t'_{k+1}$ runs over all *ordered* $(k+1)$-tuples of *distinct* integers $(t'_1, ..., t'_{k+1})$ such that $\{t'_1, ..., t'_{k+1}\} \neq \{t_1, ..., t_{k+1}\}$. Using the fact that for each ordered $(k+1)$-tuple $(t_1, ..., t_{k+1})$, there exist $(n-k-1)(n-k-2) \cdots (n-2k-1)$ $(k+1)$-tuples $(t'_1, ..., t'_{k+1})$ such that $\{t_1, ..., t_{k+1}\} \cap \{t'_1, ..., t'_{k+1}\} = \varnothing$, yielding a covariance value of zero in the above sum, and bounding $\mathrm{Cov}(\xi, \eta)$ with the arithmetic mean $\frac{1}{2}(D^2(\xi)+D^2(\eta))$ of the corresponding variances, we obtain

$$D^2_{(2)} \leqslant \frac{n(n-1)\cdot \cdots \cdot(n-k)-(n-k-1)\cdot \cdots \cdot (n-2k-1)}{[n(n-1)\cdot \cdots \cdot (n-k)]^2}$$

$$\times \sum_{1 \leqslant t_1 \neq \cdots \neq t_{k+1} \leqslant n} \cdots \sum D^2(A^{(n)}_{\mathbf{s}(\cdot), |\mathbf{Z}|}(t_1, ..., t_{k+1}) \,||\, \mathbf{Z}^{(n)}_{(\cdot)}|)$$

$$= \frac{n(n-1)\cdot \cdots \cdot(n-k)-(n-k-1)\cdot \cdots \cdot (n-2k-1)}{n(n-1)\cdot \cdots \cdot (n-k)}$$

$$\times E[D^2(A^{(n)}_{\mathbf{s}(\cdot), |\mathbf{Z}|}(R^{(n)}_{+,1}, ..., R^{(n)}_{+,k+1}) \,|\, |\mathbf{Z}^{(n)}_{(\cdot)}|)].$$

Finally,

$$E[(\Delta^{(n)})^2] = (n-k)\, E[D^2_{(1)} + D^2_{(2)}]$$

$$\leqslant \left[K + (n-k)\frac{n(n-1)\cdot \cdots \cdot(n-k)-(n-k-1)\cdot \cdots \cdot (n-2k-1)}{n(n-1)\cdot \cdots \cdot (n-k)}\right]$$

$$\times E\{[A^{(n)}_{\mathbf{s}(\cdot), |\mathbf{Z}|}(R^{(n)}_{+,1}, ..., R^{(n)}_{+,k+1})]^2\}.$$

The coefficient in this latter expression is $O(1)$, whereas, because of (2.8), the expectation converges to zero, as $n \to \infty$, under $H^{(n)}_f$. This completes the proof.

Note that $\mathscr{S}^{(n)}_+$ in general does not reduce to the same form as the statistic $\mathscr{S}^{(n)}$ appearing in Hallin *et al.* [17, formula (4.4)]. Accordingly, the asymptotic equivalence between signed-rank statistics and an unsigned one we are establishing in Section 3 below for the particular case of autocorrelation coefficients does not hold for general signed-rank statistics.

The alternatives $K^{(n)}(\mathbf{a}, \mathbf{b}; g)$ we are considering in part (ii) of Proposition 2.2 below are general linear serial dependence alternatives, under which the observed series $\mathbf{Z}^{(n)}$ is a finite realization of some stochastic process satisfying

$$X_t - n^{-1/2} \sum_{i=1}^{\infty} a_i Z_{t-i} = \varepsilon_t + n^{-1/2} \sum_{i=1}^{\infty} b_i \varepsilon_{t-i}, \qquad t \in \mathbb{Z},$$

where $\mathbf{a} = (a_1, a_2, ...)$ and $\mathbf{b} = (b_1, b_2, ...)$ are such that $\sum_{i=1}^{\infty} |a_i| < \infty$ and $\sum_{i=1}^{\infty} |b_i| < \infty$ (see, e.g., [37, Section 3.5.7]), and where $\{\varepsilon_t\}$ is an independent white noise, with density g satisfying the following assumptions:

(A1) $g(x) = g(-x)$, and $\int x^6 g(x)\, dx < \infty$.

(A2) $g(x)$ is *absolutely continuous on finite intervals* (see [16]). Hence there exists a function $\dot{g}$ such that $g(b) - g(a) = \int_a^b \dot{g}(x)\, dx$, $-\infty < a < b < \infty$, and $\dot{g}(x) = dg(x)/dx$ a.e. Defining $\varphi_g = -\dot{g}/g$, assume that $\int |\varphi_g(x)|^{2+\delta} g(x)\,dx < \infty$ for some $\delta > 0$; this implies finite Fisher information $I(g) = \int \varphi_g^2(x)\, g(x)\, dx$.

(A3) A finite derivative $\dot{\varphi}_g(x)) = d(\varphi_g(x))/dx$ exists at all but a finite number of points, and satisfies the Lipschitz condition $|\dot{\varphi}_g(x) - \dot{\varphi}(y)| < K_g\,|x-y|$ for all x, y such that $\dot{\varphi}_g(z)$ exists $\forall z \in (x, y)$.

PROPOSITION 2.2. (i) *Assume that* (2.7) *and* (2.8) *hold. Then* $(n-k)^{1/2}\,(S_+^{(n)} - m_+^{(n)})$ *is asymptotically normal under* $H_0^{(n)}$, *with mean zero and variance*

$$V_+^2 = E[(J_+^*(V_1, ..., V_{k+1}))^2] + 2 \sum_{i=1}^{k} E[J_+^*(V_1, ..., V_{k+1})\, J_+^*(V_{1+i}, ..., V_{k+1+i})]. \qquad (2.13)$$

(ii) *Provided, moreover, that the density g satisfies the technical assumptions* (A1)–(A3), $(n-k)^{1/2}\,(S_+^{(n)} - m_+^{(n)})$ *is asymptotically normal under* $K^{(n)}(\mathbf{a}, \mathbf{b}; g)$, *still with variance* V_+^2, *but with mean* $\sum_{i=1}^{k} (a_i + b_i)\, C_i^+$, *where*

$$C_i^+ = E[J_+(V_1, ..., V_{k+1}) \sum_{j=0}^{k-i} \varphi_g(G_+^{-1}(V_{1+j}))\, G_+^{-1}(V_{1+j+i})] \quad (2.14)$$

(*notation G and* G_+ *are used here in an obvious fashion*); *whether* J_+ *or* J_+^* *is used in* (2.14) *does not affect the value of* C_i^+.

Proof. Let $\mathbf{Y}_t^{(n)} = (Z_t^{(n)}, Z_{t-1}^{(n)}, ..., Z_{t-k}^{(n)})'$, $k+1 \leqslant t \leqslant n$, and consider the kernels (of order k)

$$\begin{aligned}\Psi_{\mathscr{S}}(\mathbf{y}_1, ..., \mathbf{y}_{k+1}) &= (k+1)^{-1} \sum_{j=1}^{k+1} J_+(\operatorname{sgn}(y_{j,1})(2F(|y_{j,1}|) - 1), \\ &\quad ..., \operatorname{sgn}(y_{j,k+1})(2F(|y_{j,k+1}|) - 1))\end{aligned}$$

and

$$\begin{aligned}\Psi_{\mathscr{E}}(\mathbf{y}_1, ..., \mathbf{y}_{k+1}) = [2^k(k+1)!]^{-1} \sum_{\mathscr{P}} \sum_{\mathbf{s}} J_+(s_1(2F(|y_{j,1}|) - 1), \\ ..., s_{k+1}(2F(|y_{j,k+1}|) - 1)),\end{aligned}$$

where $\mathbf{y}_j = (y_{j,1}, ..., y_{j,k+1})' \in \mathbb{R}^{k+1}$, and the summations $\sum_{\mathscr{P}}$ and $\sum_{\mathbf{s}}$ run over all $(k+1)!$ permutations $(j_1, ..., j_{k+1})$ of $(1, ..., k+1)$ and all elements $\mathbf{s} = (s_1, ..., s_{k+1})$ of $\{-1, 1\}^{k+1}$, respectively. Then, denoting by $\mathscr{U}_{\mathscr{S}}^{(n)}$ and $\mathscr{U}_{\mathscr{E}}^{(n)}$ the U-statistics induced from the $(k+1)$-dimensional series

$\mathbf{Y}_{k+1}^{(n)}, ..., \mathbf{Y}_n^{(n)}$, of length $n-k$, by the kernels $\Psi_{\mathscr{S}}$ and $\Psi_{\mathscr{E}}$, respectively, we have under $H_f^{(n)}$

$$(\mathscr{S}_+^{(n)} - \mathscr{E}_+^{(n)}) - (\mathscr{U}_{\mathscr{S}}^{(n)} - \mathscr{U}_{\mathscr{E}}^{(n)})] = o_p((n-k)^{-1/2}). \tag{2.15}$$

The proof of (2.15) follows roughly along the same lines as in Hallin *et al.* [17, Section 1.2], and is left to the reader. Being $(k+1)$-dependent, the process $\{\mathbf{Y}_t^{(n)}\}$ is, still under $H_f^{(n)}$, absolutely regular and, in view of (2.7), the conditions of Yoshihara's [42] central limit theorem for U-statistics hold for $\mathscr{U}_{\mathscr{S}}^{(n)} - \mathscr{U}_{\mathscr{E}}^{(n)}$. The g_1 function in Yoshihara's notation is here

$$\begin{aligned}
g_1(\mathbf{Y}_t^{(n)}) &= E\{\Psi_{\mathscr{S}}(\mathbf{Y}_t^{(n)}, \mathbf{Y}_{k+1}^{(n)}, ..., \mathbf{Y}_{2k}^{(n)}) - \Psi_{\mathscr{E}}(\mathbf{Y}_t^{(n)}, \mathbf{Y}_{k+1}^{(n)}, ..., \mathbf{Y}_{2k}^{(n)}) \mid Y_t^{(n)}\} \\
&= (k-1)^{-1} \Bigg[J_+(\operatorname{sgn}(Z_t^{(n)})(2F(|Z_t^{(n)}|)-1), \\
&\quad ..., \operatorname{sgn}(Z_{t-k}^{(n)})(2F(|Z_{t-k}^{(n)}|)-1)) \\
&\quad + 2^{-(k+1)} k \sum_s \int_0^1 \cdots \int_0^1 J_+(s_1 u_1, ..., s_{k+1} u_{k+1})\, du_1 \cdots du_{k+1} \\
&\quad - 2^{-(k+1)} \sum_{\mathbf{s}} \sum_{l=0}^{k} \int_0^1 \cdots \int_0^1 J_+(s_1 u_1, ..., s_l u_l, s_{l+1}(2F(|Z_t^{(n)}|)-1), \\
&\quad s_{l+2} u_{l+1}, ..., s_{k+1} u_k)\, du_1 \cdots du_k \Bigg] \\
&= (k+1)^{-1} J_+^*(\operatorname{sgn}(Z_t^{(n)})(2F(|Z_t^{(n)}|)-1), \\
&\quad ..., \operatorname{sgn}(Z_{t-k}^{(n)})(2F(|Z_{t-k}^{(n)}|)-1)) \\
&= (k+1)^{-1} J_+^*(F_+(Z_t^{(n)}), ..., F_+(Z_{t-k}^{(n)})).
\end{aligned}$$

The asymptotic normality under $H_f^{(n)}$, with mean zero and variance (2.13), of $(n-k)^{1/2}(\mathscr{U}_{\mathscr{S}}^{(n)} - \mathscr{U}_{\mathscr{E}}^{(n)})$—hence, on account of Proposition 2.1, of $(n-k)^{1/2}(S_+^{(n)} - m_+^{(n)})$ follows. Since the limit distribution does not depend on the density f, on which no assumption whatever has been made (except for symmetry), the result holds under $H_0^{(n)}$, which completes the proof of part (i) of the proposition. As for part (ii), it essentially follows along the same lines as in Hallin and Puri [22, Section 2.1], by establishing the asymptotic joint normality of $(n-k)^{1/2}(S_+^{(n)} - m_+^{(n)})$ and the log-likelihood function, then applying LeCam's third lemma (in Hájek and Šidák's [16] terminology). The covariance terms in this joint asymptotic distribution yield, under $K^{(n)}(\mathbf{a}, \mathbf{b}; g)$ (note that assumptions (A1)–(A3) on g are necessary to ensure contiguity), the desired form for the mean, although with constants

$$\begin{aligned}
C_i^+ = \sum_{j=0}^{k-i} \int_{[0,1]^{k+1}} & J_+^*(2u_1 - 1, ..., 2u_{k+1} - 1) \\
& \times \varphi_g(G^{-1}(u_{j+1}))\, G^{-1}(u_{j+i+1})\, du_1 \cdots du_{k+1}.
\end{aligned}$$

However, it follows from (2.9) and the fact that $\int_0^1 \varphi_g(G^{-1}(u))\,du = 0 = \int_0^1 G^{-1}(u)\,du$ that

$$\int_0^1 [J_+(2u_1-1, ..., 2u_{k+1}-1) - J_+^*(2u_1-1, ..., 2u_{k+1}-1)]$$
$$\times\, \varphi_g(G^{-1}(u_l))\, G^{-1}(u_{l'})\, du_1 \cdots du_{k+1} = 0,$$

for all $1 \leqslant l \neq l' \leqslant k+1$. The details are left to the reader.

3. Signed f-Rank Autocorrelations

3.1. *Definitions*

A particular type of serial rank statistic has been shown [22] to play an essential role in rank-based inference for linear time-series models: the so-called *f-rank autocorrelation coefficients*, whose role is comparable to that of ordinary parametric autocorrelations in classical, Gaussian time-series analysis. We introduce here two asymptotically equivalent definitions of a concept of *signed* f-rank autocorrelation coefficients. Denote by g a symmetric density function satisfying assumptions (A1)–(A3), by G and $\varphi_g = -\dot{g}/g$ the corresponding distribution and score functions. Let $G_+ = 2G-1$. Assume furthermore that

(A4) g is *strongly unimodal*; i.e., φ_g is a nondecreasing function.

The most convenient definition of a *signed-rank autocorrelation of order k associated with density g* is

$$r_{k;g}^{(n)+} = [(n-k)\,\sigma_{g+}^{(n)}]^{-1} \sum_{t=k+1}^{n} \operatorname{sgn}(Z_t^{(n)} Z_{t-k}^{(n)})$$
$$\times\, \varphi_g\left(G_+^{-1}\left(\frac{R_{+,t}^{(n)}}{n+1}\right)\right) G_+^{-1}\left(\frac{R_{+,t-k}^{(n)}}{n+1}\right), \tag{3.1}$$

where

$$(\sigma_{g+}^{(n)})^2 = [n(n-1)]^{-1} \sum\sum_{1 \leqslant i_1 \neq i_2 \leqslant n} \left[\varphi_g\left(G_+^{-1}\left(\frac{i_1}{n+1}\right)\right) G_+^{-1}\left(\frac{i_2}{n+1}\right)\right]^2$$
$$= [n(n-1)]^{-1} \left\{\sum_{i=1}^{n} \left[\varphi_g\left(G_+^{-1}\left(\frac{i}{n+1}\right)\right)\right]^2 \sum_{i=1}^{n} \left[G_+^{-1}\left(\frac{i}{n+1}\right)\right]^2\right.$$
$$\left. - \sum_{i=1}^{n} \left[\varphi_g\left(G_+^{-1}\left(\frac{i}{n+1}\right)\right) G_+^{-1}\left(\frac{i}{n+1}\right)\right]^2\right\} \tag{3.2}$$

denotes the exact variance (under $H_o^{(n)}$) of $(n-k)^{-1/2}$ times the sum $\sum_{t=k+1}^{n} \cdots$ in (3.1): $(n-k)^{1/2} r_{k;g}^{(n)+}$ is therefore exactly standardized under $H_0^{(n)}$. In order to see that (3.1) constitutes a serial signed-rank statistic of the form (2.1), note that, because of the symmetry of g,

$$\operatorname{sgn}(Z_t^{(n)} Z_{t-k}^{(n)})\, \varphi_g\left(G_+^{-1}\left(\frac{R_{+,t}^{(n)}}{n+1}\right)\right) G_+^{-1}\left(\frac{R_{+,t-k}^{(n)}}{n+1}\right)$$
$$= \varphi_g(G_+^{-1}(\operatorname{sgn}(Z_t^{(n)})\, R_{+,t}^{(n)}/(n+1)))\, G_+^{-1}(\operatorname{sgn}(Z_{t-k}^{(n)})\, R_{+,t-k}^{(n)}/(n+1)); \tag{3.3}$$

(3.1) is thus characterized by the scores

$$a_+^{(n)}(i_1, \ldots, i_{k+1}) = (\sigma_{g+}^{(n)})^{-1}\, \varphi_g(G_+^{-1}(i_1/(n+1)))\, G_+^{-1}(i_{k+1}/(n+1)). \tag{3.4}$$

The form of $r_{k,g}^{(n)+}$ is very similar to that of its unsigned counterpart

$$r_{k;g}^{(n)} = [(n-k)\,\sigma_{k;g}^{(n)}]^{-1}\left\{\sum_{t=k+1}^{n} \varphi_g\left(G^{-1}\left(\frac{R_t^{(n)}}{n+1}\right)\right) G^{-1}\left(\frac{R_{t-k}^{(n)}}{n+1}\right) - m_g^{(n)}\right\},$$

where $R_t^{(n)}$ denotes the (unsigned) rank of $Z_t^{(n)}$ among $Z_1^{(n)}, \ldots, Z_n^{(n)}$,

$$m_g^{(n)} = [n(n-1)]^{-1} \sum\sum_{1 \leqslant i_1 \neq i_2 \leqslant n} \varphi_g\left(G^{-1}\left(\frac{i_1}{n+1}\right)\right) G^{-1}\left(\frac{i_2}{n+1}\right),$$

and

$$\begin{aligned}
(\sigma_{k;g}^{(n)})^2 = {} & [n(n-1)]^{-1} \sum\sum_{1 \leqslant i_1 \neq i_2 \leqslant n} \left[\varphi_g\left(G^{-1}\left(\frac{i_1}{n+1}\right)\right) G^{-1}\left(\frac{i_2}{n+1}\right)\right]^2 \\
& + 2(n-2k)[n(n-1)(n-2)(n-k)]^{-1} \\
& \times \sum\sum\sum_{1 \leqslant i_1 \neq i_2 \neq i_3 \leqslant n} \left[\varphi_g\left(G^{-1}\left(\frac{i_1}{n+1}\right)\right) \varphi_g\left(G^{-1}\left(\frac{i_2}{n+1}\right)\right)\right. \\
& \left. \times G^{-1}\left(\frac{i_2}{n+1}\right) G^{-1}\left(\frac{i_3}{n+1}\right)\right] \\
& + (n^2 - n(2k+3) + k^2 + 5k)[n(n-1)(n-2)(n-3)(n-k)]^{-1} \\
& \times \sum\sum\sum\sum_{1 \leqslant i_1 \neq i_2 \neq i_3 \neq i_4 \leqslant n} \left[\varphi_g\left(G^{-1}\left(\frac{i_1}{n+1}\right)\right) \varphi_g\left(G^{-1}\left(\frac{i_2}{n+1}\right)\right)\right. \\
& \left. \times G^{-1}\left(\frac{i_3}{n+1}\right) G^{-1}\left(\frac{i_4}{n+1}\right)\right] - (n-k)(m_g^{(n)})^2;
\end{aligned}$$

$(\sigma_{k;g}^{(n)})^2$ again denotes the variance (under $H_o^{(n)}$) of $(n-k)^{-1/2}$ times the sum

$\sum_{t=k+1} \cdots$ (see [22, (2.5)]). Note that $\sigma_{g+}^{(n)}$ is considerably simpler to compute than its unsigned counterpart $\sigma_{k;g}^{(n)}$, due to the fact that all the covariance terms in (3.2) cancel out; moreover, $\sigma_{g+}^{(n)}$ does not depend on k.

Particularizing the density type, we obtain the *van der Waerden signed rank autocorrelations* (associated with normal densities)

$$r_{k;\mathrm{vdW}}^{(n)+} = [\sigma_{\mathrm{vdW}+}^{(n)}(n-k)]^{-1} \sum_{t=k+1}^{n} \operatorname{sgn}(Z_t^{(n)} Z_{t-k}^{(n)}) \times \Phi^{-1}\left(\frac{1}{2} + \frac{R_{+,t}^{(n)}}{2(n+1)}\right) \Phi^{-1}\left(\frac{1}{2} + \frac{R_{+,t-k}^{(n)}}{2(n+1)}\right), \tag{3.5}$$

where $\Phi(x) = (2\pi)^{-1/2} \int_{-\infty}^{x} \exp(-u^2/2)\, du$, the *Wilcoxon signed autocorrelations* (associated with logistic densities)

$$r_{k;\mathrm{W}}^{(n)+} = [\sigma_{\mathrm{W}+}^{(n)}(n-k)(n+1)]^{-1} \times \sum_{t=k+1}^{n} \operatorname{sgn}(Z_t^{(n)} Z_{t-k}^{(n)})\, R_{+,t}^{(n)} \log\left(\frac{n+1+R_{+,t-k}^{(n)}}{n+1-R_{+,t-k}^{(n)}}\right), \tag{3.6}$$

and, the *Laplace signed rank autocorrelations* (associated with double exponential densities)

$$r_{k;\mathrm{L}}^{(n)+} = -[\sigma_{\mathrm{L}+}^{(n)}(n-k)]^{-1} \sum_{t=k+1}^{n} \operatorname{sgn}(Z_t^{(n)} Z_{t-k}^{(n)}) \log\left(1 - \frac{R_{+,t-k}^{(n)}}{n+1}\right). \tag{3.7}$$

An inspection of formulas (3.5) through (3.7) reveals that the van der Waerden autocorrelation (3.5) is the usual autocorrelation coefficient computed from the series of the "signed standard normal quantiles" of the original series $\mathbf{Z}^{(n)}$: $\operatorname{sgn}(Z_t^{(n)})\, \Phi^{-1}(\frac{1}{2} + R_{+,t}^{(n)}/2(n+1))$ is indeed equal to $\operatorname{sgn}(Z_t^{(n)})\, \zeta_{(n+1-R_{+,t}^{(n)})/2(n+1)}$, where ζ_λ denotes the $(1-\lambda)$-quantile of the standard normal distribution. The Wilcoxon autocorrelation (3.6) is a weighted version of the usual *Wilcoxon signed rank statistic*, and the Laplace autocorrelation (3.7) is a weighted version of order k of the traditional *runs test statistic*.

An alternative definition for the signed-rank autocorrelation of order k associated with g is

$$\tilde{r}_{k;g}^{(n)+} = [(n-k)\, \tilde{\sigma}_{g+}^{(n)}]^{-1} \sum_{t=k+1}^{n} \operatorname{sgn}(Z_t^{(n)} Z_{t-k}^{(n)}) \times E\{\varphi_g(G_+^{-1}(U_n^{(R_{+,t}^{(n)})}))\}\, E\{G_+^{-1}(U_n^{(R_{+,t-k}^{(n)})})\}, \tag{3.8}$$

where $U_n^{(1)} \leqslant U_n^{(2)} \leqslant \cdots \leqslant U_n^{(n)}$ denote an ordered sample of i.i.d. rectangular [0, 1] random variables, and

$$(\tilde{\sigma}_{g+}^{(n)})^2 = [n(n-1)]^{-1} \left\{ \sum_{i=1}^{n} \{E[\varphi_g(G_+^{-1}(U_n^{(i)}))]\}^2 \sum_{i=1}^{n} \{E[G_+^{-1}(U_n^{(i)})]\}^2 - \sum_{i=1}^{n} \{E[\varphi_g(G_+^{-1}(U_n^{(i)}))]\, E[G_+^{-1}(U_n^{(i)})]\}^2 \right\}.$$

The corresponding scores are

$$\tilde{a}_+^{(n)}(i_1, ..., i_{k+1}) = (\tilde{\sigma}_{g+}^{(n)})^{-1}\, E[\varphi_g(G_+^{-1}(\mathrm{sgn}(i_1)\, U_n^{(|i_1|)})] \times E[G_+^{-1}(\mathrm{sgn}(i_{k+1})\, U^{(|i_{k+1}|)})]. \tag{3.9}$$

Particularizing (3.8) to the Gaussian case yields a *Fraser–Fisher–Yates autocorrelation coefficient,*

$$r_{k,\mathrm{FY}}^{(n)+} = [\sigma_{\mathrm{FY}}^{(n)}(n-k)]^{-1} \sum_{t=k+1}^{n} \mathrm{sgn}(Z_t^{(n)} Z_{t-k}^{(n)}) \times E[(\chi_n^{(R_{+,t}^{(n)})})^{1/2}]\, E[(\chi_n^{(R_{+,t-k}^{(n)})})^{1/2}],$$

where $\chi_n^{(1)} \leqslant \chi_n^{(2)} \leqslant \cdots \leqslant \chi_n^{(n)}$ denote an ordered sample from the chi-square distribution with one degree of freedom.

3.2. *Asymptotic Distribution of Signed-Rank Autocorrelations*

In order to justify the above definitions of signed-rank autocorrelations (as well as the claimed asymptotic equivalence of $r_{k;f}^{(n)+}$ and $\tilde{r}_{k;f}^{(n)+}$), it is sufficient to establish the asymptotic equivalence of $r_{k;f}^{(n)+}$ and $\tilde{r}_{k;f}^{(n)+}$ with their unsigned counterparts $r_{k;f}^{(n)}$.

PROPOSITION 3.1. *Assume that assumptions* (A1)–(A4) *hold. Then both* $(n-k)^{1/2}\,(r_{k;g}^{(n)+} - r_{k;g}^{(n)})$ *and* $(n-k)^{1/2}\,(\tilde{r}_{k;g}^{(n)+} - r_{k;g}^{(n)})$ *are* $o_p(1)$, *as* $n \to \infty$, *under* $H_0^{(n)}$, *and thus under any sequence of alternatives contiguous to* $H_0^{(n)}$.

Before proceeding to the proof of Proposition 3.1, we first state the following lemma, the proof of which easily follows by using a multivariate version of Hájek and Šidák's [16] proof of Lemma V.1.6a.

LEMMA 3.1. *If the function* $J_+(v_1, ..., v_{k+1})$ *is a.e. (on* $(-1, 1)^{k+1}$*) continuous, non-decreasing with respect to all its arguments, and if either*

$$a^{(n)}(i_1, ..., i_{k+1}) = J_+\left(\frac{i_1}{n+1}, ..., \frac{i_{k+1}}{n+1}\right)$$

or

$$a^{(n)}(i_1, ..., i_{k+1}) = E\{J_+(\mathrm{sgn}(i_1)\, U_n^{(|i_1|)}, ..., \mathrm{sgn}(i_{k+1})\, U_n^{(|i_{k+1}|)})\},$$

where $U_n^{(1)}, ..., U_n^{(n)}$ *denote an ordered sample of i.i.d. rectangular* $[0, 1]$ *variables, then condition* (2.8) *is satisfied.*

Proof of Proposition 3.1. Clearly,

$$\operatorname{sgn}(zz')\, \varphi_g\left(G_+^{-1}\left(\frac{i}{n+1}\right)\right) G_+^{-1}\left(\frac{i'}{n+1}\right)$$
$$= \varphi_g\left(G_+^{-1}\left(\operatorname{sgn}(z)\frac{i}{n+1}\right)\right) G_+^{-1}\left(\operatorname{sgn}(z')\frac{i'}{n+1}\right).$$

Also the strong unimodality property of g implies that φ_g is a monotonically increasing function. In view of Lemma 3.1, $\sigma_{g+}^{(n)} r_{k;g}^{(n)+}$ and $\tilde{\sigma}_{g+}^{(n)} \tilde{r}_{k;g}^{(n)+}$ thus both admit

$$\varphi_g(G_+^{-1}(v_1))\, G_+^{-1}(v_{k+1}) = \varphi_g\left(G^{-1}\left(\frac{1+v_1}{2}\right)\right) G^{-1}\left(\frac{1+v_{k+1}}{2}\right)$$

as their score-generating function. It then follows from (2.8) that

$$\lim_{n\to\infty} (\sigma_{g+}^{(n)})^2 = \lim_{n\to\infty} (\tilde{\sigma}_{g+}^{(n)})^2$$
$$= \int_0^1 \int_0^1 [\varphi_g(G^{-1}(u_1))\, G^{-1}(u_2)]^2\, du_1 du_2 = \sigma_g^2 I(g),$$

where σ_g^2 stands for $\int x^2 g\, dx$. Accordingly, $r_{k;g}^{(n)+}$ and $\tilde{r}_{k;g}^{(n)+}$ admit

$$J_+(v_1, ..., v_{k+1}) = [\sigma_g^2 I(g)]^{-1/2}\, \varphi_g\left(G^{-1}\left(\frac{1+v_1}{2}\right)\right) G^{-1}\left(\frac{1+v_{k+1}}{2}\right)$$

as their score-generating function. Definitions (2.10) and (2.11) then yield (under $H_f^{(n)}$)

$$\mathscr{S}_+^{(n)} = [(n-k)\, \sigma_g^2 I(f)]^{-1} \sum_{t=k+1}^{n} \varphi_g(G^{-1}(F(Z_t^{(n)})))\, G^{-1}(F(Z_{t-k}^{(n)})), \tag{3.10}$$

and $\mathscr{E}_+^{(n)} = 0$. Now (3.10) exactly coincides with the $\mathscr{S}^{(n)}$ function associated with $r_{k;g}^{(n)}$ (cf. [18, Proposition 3.1]), whereas the corresponding $\mathscr{E}^{(n)}$ is [17, formula (4.4)])

$$\mathscr{E}^{(n)} = [n(n-1)]^{-1} \sum_{t_1 \neq t_2}\sum \varphi_g(G^{-1}(F(Z_{t_1}^{(n)})))\, G^{-1}(F(Z_{t_2}^{(n)}))$$
$$= n^{-2} \sum_{t=1}^{n} \varphi_g(G^{-1}(F(Z_t^{(n)}))) \sum_{t=1}^{n} G^{-1}(F(Z_t^{(n)})) + o_p(n^{-1/2})$$
$$= o_p(n^{-1/2}),$$

since

$$n^{-1} \sum \varphi_g(G^{-1}(F(Z_t^{(n)}))) \xrightarrow{P} \int \varphi_g(G^{-1}(u))\, du = 0$$

and

$$n^{-1/2} \sum G^{-1}(F(Z_t^{(n)})) \xrightarrow{\mathscr{L}} N(0, \sigma_g),$$

as $n \to \infty$. The proposition then follows from the fact that $r_{k;g}^{(n)} - (\mathscr{S}^{(n)} - \mathscr{E}^{(n)})$ is $o_p(n^{-1/2})$.

4. Locally Asymptotically Optimal Signed-Rank Tests for ARMA Models

4.1. *Locally Asymptotically Maximin Tests*

Testing the adequacy of an ARMA model is certainly one of the most fundamental testing problems in time-series analysis, both in its own right and because of its implications in the identification (see e.g. [35]) and validation steps of the "Box–Jenkins methodology." We show here how signed-rank statistics provide locally asymptotically optimal tests (in the maximin sense) for this important problem, when the innovation density under the null hypothesis remains unspecified (but symmetric).

Let us denote by $H^{(n)}(\mathbf{A}, \mathbf{B}; \cdot)$ (resp. $H^{(n)}(\mathbf{A}, \mathbf{B}; f)$) the hypothesis under which an observed series $\mathbf{X}^{(n)} = (X_{-p_1+1}^{(n)}, ..., X_0^{(n)}, X_1^{(n)}, ..., X_n^{(n)})$ is generated by the ARMA (p_1, q_1) model

$$X_t - \sum_{i=1}^{p_1} A_i X_{t-i} = \varepsilon_t + \sum_{i=1}^{q_1} B_i \varepsilon_{t-i}, \qquad t \in \mathbb{Z}, \tag{4.1}$$

where $\{\varepsilon_t\}$ is an i.i.d. innovation process with unspecified symmetric density (resp. with symmetric density f satisfying assumptions (A1)–(A4)). The polynomials $A(z) = 1 - \sum_{i=1}^{p_1} A_i z^i$ and $B(z) = 1 + \sum_{i=1}^{q_1} B_i z^i$, $z \in \mathbb{C}$, are required to have distinct roots, all lying outside the unit-circle, so that the usual stationarity and invertibility conditions be satisfied.

In order to obtain local alternatives, consider the sequence of ARMA (p_2, q_2) models $(p_2 \geqslant p_1,\ q_2 \geqslant q_1)$

$$\begin{aligned} X_t - \sum_{i=1}^{p_1} (A_i + n^{-1/2}\gamma_i)\, X_{t-i} - n^{-1/2} \sum_{i=p_1+1}^{p_2} \gamma_i X_{t-i} \\ = \varepsilon_t + \sum_{i=1}^{q_1} (B_i + n^{-1/2}\delta_i)\, \varepsilon_{t-i} + n^{-1/2} \sum_{i=q_1+1}^{q_2} \delta_i \varepsilon_{t-i}, \qquad t \in \mathbb{Z}, \end{aligned} \tag{4.2}$$

where $\{\varepsilon_t\}$ again denotes an i.i.d. innovation process with density f

satisfying assumptions (A1)–(A4). Denote by $K^{(n)}(\mathbf{A}+\boldsymbol{\gamma}, \mathbf{B}+\boldsymbol{\delta}; f)$ the hypothesis under which $\mathbf{X}^{(n)}$ is generated by (4.2).

Let $\mathbf{a}=(a_1, a_2, ..., a_i, ...)$ and $\mathbf{b}=(b_1, b_2, ..., b_i, ...)$ be defined by the polynomial ratios

$$\left(\sum_{i=1}^{p_2} \gamma_i z^i\right) \bigg/ \left(1-\sum_{i=1}^{p_1} A_i z^i\right) = -\sum_{i=1}^{\infty} a_i z^i$$

and

$$\left(\sum_{i=1}^{q_2} \delta_i z^i\right) \bigg/ \left(1+\sum_{i=1}^{q_1} B_i z^i\right) = \sum_{i=1}^{\infty} b_i z^i$$

($\mathbf{a}$ and $\mathbf{b}$ also can be expressed in terms of $\boldsymbol{\gamma}$, $\boldsymbol{\delta}$, and the Green's functions associated with the difference operators $A(L)$ and $B(L)$, where L denotes the lag operator—see [22, Section 3.1]). It follows from the stationarity and invertibility properties of (4.1) that the sequences $\mathbf{a}$ and $\mathbf{b}$ are absolutely summable: denote by $\|\mathbf{a}+\mathbf{b}\| = [\sum_{i=1}^{\infty} (a_i+b_i)^2]^{1/2}$, the l^2 norm of $\mathbf{a}+\mathbf{b}$.

The power of the most powerful test (viz. the *Neyman test*) for $H^{(n)}(\mathbf{A}, \mathbf{B}; f)$ against $K^{(n)}(\mathbf{A}+\boldsymbol{\gamma}, \mathbf{B}+\boldsymbol{\delta}; f)$ at level α can be shown [22] to converge to the asymptotic value

$$1-\Phi(k_{1-\alpha} - \|\mathbf{a}+\mathbf{b}\| [\sigma^2 I(f)]^{1/2}), \tag{4.3}$$

where $\Phi(\cdot)$ and $k_{1-\alpha}$ denote the standard normal distribution function and $(1-\alpha)$-quantile, respectively. The norm $\|\mathbf{a}+\mathbf{b}\|$ thus appears as a natural distance between the sequences $H^{(n)}(\mathbf{A}, \mathbf{B}; \cdot)$ and $K^{(n)}(\mathbf{A}+\boldsymbol{\gamma}, \mathbf{B}+\boldsymbol{\delta}; f)$, ($H^{(n)}(\mathbf{A}, \mathbf{B}; f)$ actually is a least favorable distribution in the problem of testing $H^{(n)}(\mathbf{A}, \mathbf{B}; \cdot)$ against $K^{(n)}(\mathbf{A}+\boldsymbol{\gamma}, \mathbf{B}+\boldsymbol{\delta}; f)$). Accordingly, if optimal tests are to be derived against alternatives under which $\boldsymbol{\gamma} \in \mathbb{R}^{p_2}$ and $\boldsymbol{\delta} \in \mathbb{R}^{q_2}$ remain unspecified, the most natural idea is to look for an *asymptotical maximin property* against local alternatives that remain "bounded away from the null hypothesis" in the sense of the above distance. Consider therefore the *unspecified* ARMA (p_2, q_2) alternative

$$\begin{aligned} &K^{(n)}(\mathbf{A}+\mathbb{R}^{p_2}, \mathbf{B}+\mathbb{R}^{q_2}; f \mid d) \\ &\quad = \bigcup \{K^{(n)}(\mathbf{A}+\boldsymbol{\gamma}, \mathbf{B}+\boldsymbol{\delta}; f) \mid \boldsymbol{\gamma} \in \mathbb{R}^{p_2}, \boldsymbol{\delta} \in \mathbb{R}^{q_2}, \|\mathbf{a}+\mathbf{b}\| \geqslant d\}, \end{aligned} \tag{4.4}$$

where $d>0$ is some arbitrary positive constant.

Finally, denote by $\mathbf{Z}^{(n)}$ the filtered series obtained from $\mathbf{X}^{(n)}$ by applying the linear recursion

$$Z_t^{(n)} = X_t^{(n)} - \sum_{i=1}^{p_1} A_i X_{t-i}^{(n)} - \sum_{i=1}^{q_1} B_i Z_{t-i}^{(n)}, \qquad t=1, ..., n,$$

with initial values $Z_0^{(n)} = Z_{-1}^{(n)} = \cdots = Z_{-q_1+1}^{(n)} = 0$: $\mathbf{Z}^{(n)}$ is the series from which the ranks $R_{+,t}^{(n)}$ and the signs $\operatorname{sgn}(Z_t^{(n)})$ in the following proposition are to be computed.

PROPOSITION 4.1. *Let $\{\psi_t^{(1)}, \psi_t^{(2)}, ..., \psi_t^{(p_1+q_1)}\}$ be an arbitrary fundamental system of solutions of the homogeneous difference equation*

$$\psi_t - \sum_{j=1}^{p_1+q_1} \sum_{i=\max(0,j-q_1)}^{\min(j,p_1)} A_i B_{j-i} \psi_{t-j} = 0, \qquad t \in \mathbb{Z}, \tag{4.5}$$

with the convention $A_0 = -1$, $B_0 = 1$. Putting $\pi = \max(p_2 - p_1, q_2 - q_1)$, denote by $\mathbf{W}_\psi^{(n)}$ the (non-singular) $(p_1+q_1)\times(p_1+q_1)$ matrix with elements

$$W_{\psi;k,l}^{(n)} = \sum_{i=\pi+1}^{n-1} \psi_i^{(k)} \psi_i^{(l)}, \qquad k, l = 1, ..., p_1+q_1, \tag{4.6}$$

and consider the (p_1+q_1)-dimensional vector of signed-rank statistics

$$n^{1/2}\mathbf{T}_{\psi;f}^{(n)+} = \begin{pmatrix} \sum_{i=\pi+1}^{n-1} (n-i)^{1/2}\, \psi_i^{(1)} r_{i;f}^{(n)+} \\ \vdots \\ \sum_{i=\pi+1}^{n-1} (n-i)^{1/2}\, \psi_i^{(p_1+q_1)} r_{i;f}^{(n)+} \end{pmatrix}. \tag{4.7}$$

Then, the quadratic test statistic

$$Q_{A,B;f}^{(n)+} = \sum_{i=1}^{\pi} (n-i)(r_{i;f}^{(n)+})^2 + n(\mathbf{T}_{\psi;f}^{(n)+})' (\mathbf{W}_\psi^{(n)})^{-1} (\mathbf{T}_{\psi;f}^{(n)+}) \tag{4.8}$$

(i) *does not depend on the particular fundamental system $\{\psi_i^{(1)}, ..., \psi_i^{(p_1+q_1)}\}$ adopted but only on the coefficients A_i and B_i of the tested model* (4.1) (*hence the notation $Q_{A,B;f}^{(n)+}$*);

(ii) *is asymptotically chi-square, with* $\max(p_1+q_2, p_2+q_1)$ *degrees of freedom under $H^{(n)}(\mathbf{A}, \mathbf{B}; \cdot)$*;

(iii) *is asymptotically noncentral chi-square, with* $\max(p_1+q_2, p_2+q_1)$ *degrees of freedom and noncentrality parameter*

$$\tfrac{1}{2}\,\|\mathbf{a}+\mathbf{b}\|^2 \left[\int \phi(F^{-1}(u))\,\phi_g(G^{-1}(u))\,du \int F^{-1}(v)\,G^{-1}(v)\,dv\right]^2 [\sigma^2 I(f)]^{-1}$$

under $K^{(n)}(\mathbf{A}+\boldsymbol{\gamma}, \mathbf{B}+\boldsymbol{\delta}; g)$ (g here denotes an arbitrary symmetric density, satisfying assumptions (A1)–(A4); *the notation G and ϕ_g are used in an obvious fashion*);

(iv) *provides an asymptotically maximin most powerful test for $H^{(n)}(\mathbf{A}, \mathbf{B}; \cdot)$ against the unspecified ARMA (p_2, q_2) alternative* (4.4), *for* *any* $d>0$; *the corresponding envelope power function accordingly converges to* $1-F_{\max(p_1+q_2,\,p_2+q_1)}(\chi^2_{1-\alpha};\, d^2\sigma^2 I(f)/2)$, *where* $\chi^2_{1-\alpha}$ *is the* $(1-\alpha)$*-quantile of the chi-square distribution with* $\max(p_1+q_2, p_2+q_1)$ *degrees of freedom, and* $F_v(\cdot;\lambda)$ *denotes the noncentral chi-square distribution function with* v *degrees of freedom and noncentrality parameter* λ.

Proof. Since $\{\psi_t^{(1)}, ..., \psi_t^{(p_1,q_1)}\}$ is a fundamental system of solutions of (4.2), its p_1+q_1 elements are linearly independent, which ensures the nonsingularity of $\mathbf{W}_\psi^{(n)}$. Now, because of the vector structure of the space of solutions of (4.2), any fundamental system of solutions $\{\tau_t^{(1)}, ..., \tau_t^{(p_1+q_1)}\}$ is of the form

$$(\tau_t^{(1)}, ..., \tau_t^{(p_1+q_1)})' = \mathbf{K}(\psi_t^{(1)}, ..., \psi_t^{(p_1+q_1)})',$$

with $\mathbf{K}$ a full-rank $(p_1+q_1)\times(p_1+q_1)$ matrix of constants. Using obvious notation, we have

$$\mathbf{W}_\tau^{(n)} = \begin{pmatrix} \tau_{\pi+1}^{(1)} & \cdots & \tau_{n-1}^{(1)} \\ \vdots & & \vdots \\ \tau_{\pi+1}^{(k)} & \cdots & \tau_{n-1}^{(k)} \\ \vdots & & \vdots \\ \tau_{\pi+1}^{(p_1+q_1)} & \cdots & \tau_{n-1}^{(p_1+q_1)} \end{pmatrix} \begin{pmatrix} \tau_{\pi+1}^{(1)} & \cdots & \tau_{n-1}^{(1)} \\ \vdots & & \vdots \\ \tau_{\pi+1}^{(k)} & \cdots & \tau_{n-1}^{(k)} \\ \vdots & & \vdots \\ \tau_{\pi+1}^{(p_1+q_1)} & \cdots & \tau_{n-1}^{(p_1+q_1)} \end{pmatrix}'$$

$$= \mathbf{K} \begin{pmatrix} \psi_{\pi+1}^{(1)} & \cdots & \psi_{n-1}^{(1)} \\ \vdots & & \vdots \\ \psi_{\pi+1}^{(k)} & \cdots & \psi_{n-1}^{(k)} \\ \vdots & & \vdots \\ \psi_{\pi+1}^{(p_1+q_1)} & \cdots & \psi_{n-1}^{(p_1+q_1)} \end{pmatrix} \begin{pmatrix} \psi_{\pi+1}^{(1)} & \cdots & \psi_{n-1}^{(1)} \\ \vdots & & \vdots \\ \psi_{\pi+1}^{(k)} & \cdots & \psi_{n-1}^{(k)} \\ \vdots & & \vdots \\ \psi_{\pi+1}^{(p_1+q_1)} & \cdots & \psi_{n-1}^{(p_1+q_1)} \end{pmatrix}' \mathbf{K}'$$

$$= \mathbf{K}\mathbf{W}_\psi \mathbf{K}'.$$

Hence

$$\begin{aligned} n(\mathbf{T}_{\tau;f}^{(n)+})' (\mathbf{W}_\tau^{(n)})^{-1} (\mathbf{T}_{\tau;f}^{(n)+}) &= n(\mathbf{T}_{\psi;f}^{(n)+})' \mathbf{K}'(\mathbf{K}')^{-1} (\mathbf{W}_\psi^{(n)})^{-1} \mathbf{K}^{-1}\mathbf{K}(\mathbf{T}_{\psi;f}^{(n)+}) \\ &= n(\mathbf{T}_{\psi;f}^{(n)+})' (\mathbf{W}_\psi^{(n)})^{-1} (\mathbf{T}_{\psi;f}^{(n)+}), \end{aligned}$$

which completes part (i) of the proof. The other parts of the Proposition follow from the fact that, because of Proposition 3.1 (ii),

$$\Bigg[(n-1)^{1/2} r_{1;f}^{(n)+}, ..., (n-\pi)^{1/2} r_{\pi;f}^{(n)+}, \sum_{i=\pi+1}^{n-1} (n-i)^{1/2} \psi_i^{(1)} r_{i;f}^{(n)+}, ...,$$

$$\sum_{i=\pi+1}^{n-1} (n-i)^{1/2} \psi_i^{(p_1+q_1)} r_{i;f}^{(n)+}\Bigg]' - n^{1/2}\mathbf{T}_{\psi;f}^{(n)} = o_p(1),$$

under $H^{(n)}(\mathbf{A}, \mathbf{B}; \cdot)$ as well as under (4.4), where $\mathbf{T}^{(n)}_{\psi;f}$ is the rank-based statistic considered in Hallin and Puri [22, Proposition 3.2], whereas, due to the stationarity and invertibility properties of (4.1), $\mathbf{W}^{(n)}_{\psi}$ converges componentwise to the matrix $\mathbf{W}$ with elements $\sum_{i=\pi+1}^{\infty} \psi_i^{(k)} \psi_i^{(l)}$. Parts (ii), (iii), and (iv) of the proposition then follow from Proposition 4.2 and 4.3 in Hallin and Puri [22].

Note that an asymptotical form of the optimal test of the above Proposition consists in rejecting $H^{(n)}(\mathbf{A}, \mathbf{B}; \cdot)$ whenever $Q^{(n)+}_{A,B;f}$ is larger than the $(1-\alpha)$-quantile of the chi-square distribution with $\max(p_1+q_2, p_2+q_1)$ degrees of freedom. Note also that the test statistic $Q^{(n)+}_{A,B;f}$ decomposes into two asymptotically independent parts. The first one, as the sum of squared signed-rank autocorrelation coefficients of orders 1 through π, is a signed-rank version of the well-known Ljung–Box–Pierce *portmanteau* statistic [3, 29], and does not depend on the tested model (4,1). The second part, on the other hand, crucially depends on the coefficients A_i and B_i of (4.1) (through the fundamental system of solutions), and constitutes a *weighted*, signed-rank version of the portmanteau statistic.

All of the above results still hold, of course, without any modification, if the coefficients $\tilde{r}^{(n)+}_{i;f}$ are substituted for the $r^{(n)+}_{i;f}$ ones.

4.2. *Asymptotic Relative Efficiencies*

The parametric version of the problem treated in this paper (viz. that of obtaining locally optimal tests for general, possibly non-Gaussian, time-series models) has not been explicitly considered so far in the literature. The existing results deal with special cases, such as testing a specified Gaussian $AR(1)$ model [9], testing for randomness [22], testing against simple pth order autoregressive disturbances [11], testing general one-parameter Gaussian models [27], testing against additional AR or MA roots in Gaussian models [24], testing the adequacy of fitted Gaussian ARMA (p, q) models ([13, 25, 26, 33, 34]—see also [35, 36]).

It is, however, easy to see that the optimal (in the sense of Proposition 4.1) normal-theory procedure here can be obtained by substituting classical autocorrelation coefficients

$$r_k^{(n)} = n \sum_{t=k+1}^{n} Z_t^{(n)} Z_{t-k}^{(n)} / (n-k) \sum_{t=1}^{n} (Z_t^{(n)})^2 \tag{4.9}$$

for the signed-rank ones in definitions (4.7) and (4.8). The resulting quadratic test statistic $Q^{(n)}_{A,B}$ (coinciding with the appropriate Gaussian Lagrange multiplier test statistic) is then asymptotically equal (up to $o_P(1)$ terms) to the corresponding van der Waerden or Fisher–Yates–Fraser statistic, for Gaussian innovation densities, both under the null hypothesis as well as under contiguous Gaussian alternatives. Proposition 4.1 (iii) applies to this parametric test, but the noncentrality parameter under the alternative $K^{(n)}(A+\gamma, B+\delta; g)$ now is simply $\|a+b\|^2/2$ (this is a consequence of [17, Proposition 5.2]).

The mutual asymptotic relative efficiencies (AREs) of the various tests just described can be obtained as the ratios of their respective noncentrality parameters. Some numerical values, under Gaussian, logistic, and double exponential densities, respectively, are provided in Table I.

The results of Sections 3 and 4 also can be used to derive asymptotic normality and asymptotic relative efficiency results for the *signed Spearman autocorrelations*

$$r_{k;S}^{(n)+} = (n-k)^{-1} \sum_{t=k+1}^{n} \operatorname{sgn}(Z_t^{(n)} Z_{t-k}^{(n)})\, R_{+,t}^{(n)} R_{+,t-k}^{(n)} / \sigma_{+;S}^{(n)},$$

where

$$\begin{aligned}(\sigma_{+;S}^{(n)})^2 &= [n(n-1)]^{-1} \sum_{1 \leqslant i_1 \neq i_2 \leqslant n} \sum (i_1 i_2)^2 \\ &= [n(n-1)]^{-1} \left[\left(\sum_{i=1}^{n} i^2 \right)^2 - \sum_{i=1}^{n} i^4 \right] \\ &= (n+1)(20n^3 + 24n^2 - 5n - 6)/180\end{aligned}$$

TABLE I

Mutual AREs for the Various Quadratic Parametric and Rank-Based Tests Described Above—Under Gaussian, Logistic, and Double Exponential Densities, Respectively

	Classical parametric	van der Waerden–Fisher–Yates	Wilcoxon	Laplace	
Classical parametric	1.000	1.000	1.005	1.634	Normal
	1.000	0.954	0.911	1.232	Logistic
	1.000	0.816	0.675	0.500	Double exp.
van der Waerden–Fisher–Yates	1.000	1.000	1.055	1.634	Normal
	1.048	1.000	0.954	1.291	Logistic
	1.226	1.000	0.827	0.613	Double exp.
Wilcoxon	0.948	0.948	1.000	1.550	Normal
	1.098	1.048	1.000	1.352	Logistic
	1.482	1.209	1.000	0.741	Double exp.
Laplace	0.612	0.612	0.646	1.000	Normal
	0.812	0.775	0.740	1.000	Logistic
	2.000	1.631	1.350	1.000	Double exp.

TABLE II

AREs of Tests Based on Signed Spearman Autocorrelations or Runs with Respect to Their Parametric Counterparts Based on Classical Autocorrelation Coefficients, under Gaussian, Logistic, and Double Exponential Densities, Respectively

	Normal	Logistic	Double exponential
Signed Spearman	0.912	1.000	1.266
runs	0.405	0.438	0.500

is such that $(n-k)^{1/2}\, r_{k;S}^{(n)+}$ is exactly standardized. Just as signed f-rank autocorrelation coefficients, the signed Spearman ones can be shown to be asymptotically standard normal (viz. $(n-k)^{1/2}\, r_{k;S}^{(n)+} \xrightarrow{\mathscr{L}} \mathcal{N}(0, 1)$, under $H_0^{(n)}$). Asymptotic relative efficiencies of tests based on signed Spearman autocorrelations, with respect, e.g., to the corresponding parametric ones, based on ordinary sample autocorrelations, again follow from applying Proposition 2.2. Numerical values are given in Table II.

As another application of the results obtained in Sections 3 and 4, consider the kth-order generalization of the classical runs test statistic

$$S_k^{(n)} = (n-k)^{-1} \sum_{t=k+1}^{n} I[Z_t^{(n)} Z_{t-k}^{(n)} < 0], \qquad k = 1, ..., n-1, \quad (4.10)$$

where $I[\cdot]$ denotes the indicator of the set $[\cdot]$. This statistic has been introduced in Dufour and Hallin [8], where portmanteau and optimal quadratic runs tests of the form (4.8) are studied in some detail. The main advantage of such generalized runs statistics is that they remain invariant under the extremely general null hypothesis that $\mathbf{Z}^{(n)}$ is a vector of non-homogeneous (i.e., possibly nonidentically distributed), independent variables with zero median (even discrete distributions are allowed). This allows for heteroskedasticity, discrete distributions, ties, etc., making (4.10) very attractive in such fields as econometrics, for example. Though the resulting tests belong to the category of "easy and quick" tests, their asymptotic efficiencies with respect to the parametric procedures based on $r_k^{(n)}$ are not bad at all. The ARE values shown in Table II follow as an application of Proposition 2.2.

5. Examples

Testing the hypothesis that an observed series $\mathbf{X}^{(n)}$ was generated by an AR(1) model $X_t - \rho X_{t-1} = \varepsilon_t$ with autoregressive coefficient $\rho = \rho_o \in (-1, 1)$ and unspecified, symmetric innovation density against an alternative with $\rho > \rho_o$ and innovation density f can be achieved (in a locally most powerful manner, provided that f satisfies the adequate technical assumptions A1–A3) by rejecting $\rho = \rho_o$ whenever the signed-rank statistic

$$(1-\rho_0^2)^{1/2} \sum_{i=1}^{n-1} (n-i)^{1/2} \rho_o^{i-1} r_{i;f}^{(n)+} \tag{5.1}$$

exceeds the $(1-\alpha)$-standard normal quantile. The ranks to be considered in (5.1) are those of the residuals $Z_t^{(n)} = X_t^{(n)} - \rho_o X_{t-1}^{(n)}$, $t = 1, ..., n$. Spearman autocorrelations $r_{i;S}^{(n)+}$ also can be used in (5.1), either if no particular innovation density is to be emphasized under the alternative, or if particular attention is devoted to alternatives under which the technical assumptions A1–A3 are not satisfied: e.g., Cauchy or stable innovation densities.

Much attention has been devoted to processes with heavy-tailed innovation densities: see Fama [12], Mandelbrot [30], McCullough [31], among many others, for economic and financial applications. It is well known that classical results on the asymptotic distributions of usual autocorrelation coefficients do not hold anymore if the innovation variance is infinite (see, e.g., [5, 4].) Classical identification and diagnostic checking techniques, based on correlograms, consequently cannot be used safely in this context and can be totally misleading if this possibility of an infinite variance is overlooked: numerical investigation actually suggests that they behave extremely poorly. This is in very sharp contrast with our rank-based procedures, which remain perfectly valid and quite powerful—though, due to the lack of contiguity results, their local powers cannot be obtained explicitly.

In order to illustrate this point, an artificial series of length 16 has been generated from the AR(1) model $X_t - 0.65X_{t-1} = \varepsilon_t$, where the ε_t's are i.i.d. with Cauchy density. The resulting series is shown in Table III, along with the residuals $Z_t = X_t - 0.5X_{t-1}$, and the signed ranks $\text{sgn}(Z_t)\, R_{+,t}^{(15)}$ required for testing $H_0 : \rho = 0.5$. Note the significant trough starting with X_{13}—a behavior very typical in Cauchy series.

The locally optimal parametric statistic

$$(1-(0.5)^2)^{1/2} \sum_i (15-i)^{1/2} (0.5)^{i-1} r_i^{(15)},$$

where $r_i^{(15)}$ denotes the usual autocorrelation coefficient, its unsigned and signed rank-based counterparts (of the form

$$(1-(0.5)^2)^{1/2} \sum (15-i)^{1/2} (0.5)^{i-1} r_{i;s}^{(15)+},$$

TABLE III

Sixteen Observations of the AR(1) Process $X_t - 0.65X_{t-1} = \varepsilon_t$, ε_t i.i.d. with Cauchy Density, along with the Residuals $Z_t = X_t - 0.5X_{t-1}$ and the Signed Ranks $\text{sgn}(Z_t)\, R_{+,t}^{(15)}$

t	X_t	Z_t	$\text{sgn}(Z_t)\, R_{+,t}$
0	0.949123		
1	0.280853	−0.193708	−2
2	0.814711	0.674284	7
3	0.837240	0.429885	4
4	−0.826629	−1.245249	−10
5	−0.847275	−0.433961	−5
6	−1.952889	−1.529251	−11
7	−1.933655	−0.957211	−8
8	−1.986299	−1.019471	−9
9	−1.495813	−0.502664	−6
10	−0.949718	−0.201811	−3
11	−2.440600	−1.965651	−12
12	−1.132867	0.087388	1
13	−68.528602	−67.962169	−15
14	−46.126992	−11.862691	−14
15	−33.056494	−9.992998	−13

e.g., in the signed Spearman case) have been computed for this residual series Z_t, $t = 1, ..., 15$, yielding the figures shown in Table IV.

An inspection of Table IV shows how unreliable classical techniques are in the presence of Cauchy innovations: if considered in its finite-variance-theory distribution, the parametric statistic is totally inefficient. Quite on the contrary, in spite of the short series length, all rank-based statistics significantly reject the null hypothesis (except for the unsigned Laplace one). Finally, signed-rank statistics appear to be substantially more powerful here than their unsigned counterparts (they all reject at the $\alpha = 1\%$ level).

Rank-based tests thus seem to bring the best alternative to normal theory parametric techniques in the presence of infinite innovation variance.

The same hypothesis ($H_0: \rho = \rho_o$ in the AR(1) model $X_t - \rho X_{t-1} = \varepsilon_t$) could be tested against a more general alternative of ARMA (p, q) dependence

$$X_t - A_1 X_{t-1} - A_2 X_{t-2} = \varepsilon_t + B\varepsilon_{t-1}$$

with orders $1 \leqslant p \leqslant 2$ and $q \leqslant 1$ (viz. $p = 2$ if $A_2 \neq 0$, $p = 1$ if $A_2 = 0$ and

TABLE IV

Observed Values of the Signed and Unsigned Locally Optimal Spearman, van der Waerden, Wilcoxon, and Laplace Test Statistics, along with the Corresponding P-values, Computed from the Residual Series $Z_1 \cdots Z_{15}$ in Table III

	unsigned		signed	
Test statistic	Observed value	P-value	Observed value	P-value
Spearman	1.388062	0.0826	3.018533	0.0013
van der Waerden	1.674870	0.0470	2.432446	0.0075
Wilcoxon	1.690271	0.0453	2.839421	0.0023
Laplace	0.976343	0.1648	3.789278	0.0003
Parametric	0.294180	0.3844		

Note. The observed value and P-value of the parametric normal theory test statistic are also provided—though the normal approximation is not valid: this latter P-value thus is not correct. All P-values follow from normal approximation.

$A_1 \neq 0$; $q=1$ if $B \neq 0$, $q=0$ if $B=0$) and unspecified coefficients A_1, A_2, and B. For $\rho_o = 0.5$, the test statistic to be used would be of the form

$$(n-1)(r_{1;f}^{(n)+})^2 + \left[\sum_{i=2}^{n-1} (n-i)^{1/2} \rho_o^{i-2} r_{i;f}^{(n)+}\right]^2 (1-\rho_o^2), \qquad (5.2)$$

where $\rho_o = 0.5$ and the ranks still are those shown in Table III. The observed value of (5.2) should be compared with the $(1-\alpha)$-quantile of a chi-square variable with two degrees of freedom. If a Spearman version of (5.2) is to be considered, the resulting ARE, with respect to the corresponding Gaussian Lagrange multiplier test (see [26]) is 0.912 under Gaussian innovations, but 1.000 under logistic and 1.266 under double-exponential ones (see Table II).

The numerical values of the signed Spearman statistic

$$14(r_{1;S}^{(15)+})^2 + \left[\sum_{i \geqslant 2} (15-i)^{1/2} (0.5)^{i-2} r_{i;S}^{(15)+}\right]^2 (1-(0.5)^2),$$

its unsigned and Gaussian parametric counterparts are shown in Table V, along with their P-values.

All the previous remarks remain valid here. The figures in Table V show that the tests based on (5.2) are not as powerful (against $A_1 = 0.65$, $A_2 = B_1 = 0$) as those based on (5.1); this is not very surprising, since (5.1) is locally optimal against this type of alternative, whereas (5.2) is not.

TABLE V

Observed Values, and P-values, of the Signed Spearman, Unsigned Spearman, and Parametric Statistics (5.2), for the Series Given in Table III

	Observed value	P-value
Signed Spearman	4.2491	0.1195
Unsigned Spearman	3.2779	0.1942
Parametric	0.1250	0.9394

Note. P-values are derived from asymptotic chi-square approximation (2 of freedom).

A more systematic Monte Carlo study of the advantage of signed-rank techniques over unsigned and parametric ones can be found in Hallin *et al.* [21].

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